

Differentiation

"Axioms" of Continuity. Let

$-\infty < a < b < +\infty$, and let the real valued function f be continuous on $[a, b]$. Then,

1. As x runs between a and b , inclusive of a and b , $f(x)$ takes on each value between $f(a)$ and $f(b)$, inclusive of $f(a)$ and $f(b)$, at least once. (Intermediate value theorem.)

2. Each of

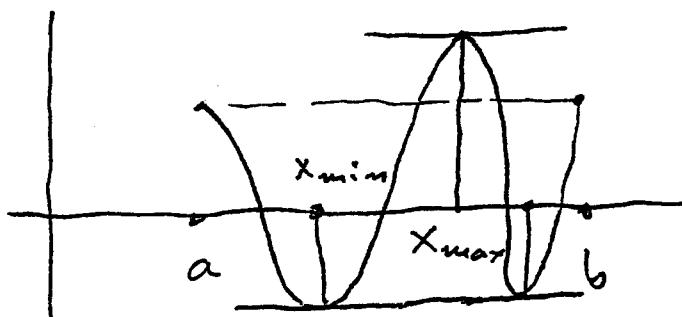
$$\min \{f(x) : a \leq x \leq b\}$$

and

$$\max \{f(x) : a \leq x \leq b\}$$

is attained, at least once, as x runs from a to b , inclusive.

□



Picture of Rolle's theorem



Let the real valued f be defined on (a, b) , and let x be in (a, b) . f is differentiable at x if

$$\frac{f(x+h) - f(x)}{h} \rightarrow f'(x), \quad h \rightarrow 0, \quad (1)$$

that is, if

$$r(h) := \frac{f(x+h) - f(x)}{h} - f'(x) \rightarrow 0, \quad h \rightarrow 0,$$

that is, if

$$f(x+h) = f(x) + f'(x)h + r(h)h,$$

with (2)

$$r(h) \rightarrow 0, \text{ as } h \rightarrow 0.$$

Note that these two statements are equivalent: if (2) holds then so does (1)! And (2) will prove to be the best way to think of differentiation. Replacing $x+h$ by x and x by c gives

$$f(x) = f(c) + f'(c)(x-c) + r(x-c)(x-c)$$

with

$$r(x-c) \rightarrow 0 \text{ as } x \rightarrow c.$$

The straight line

$$l(c) : y = f(c) + f'(c)(x - c)$$

is the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$ on the curve. It is the linearization of f about the point $x = c$. It is a pretty good approximation to $f(x)$, if x is near c . The remainder term $r(h)h$, $h := x - c$, is the error in this approximation.

Examples. 1. $f(x) = x^2$, x real

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= 2x + h \end{aligned}$$

$$\rightarrow 2x, h \rightarrow 0$$

2. $f(x) = \sqrt{x}$, $x \geq 0$

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$\begin{aligned}
 &= \frac{(x+h) - x}{h} \cdot \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
 &\rightarrow \frac{1}{2\sqrt{x}}, \quad h \rightarrow 0, \quad \underline{\underline{x > 0}}
 \end{aligned}$$

Using our "linearization" idea of the derivative it's easy to give a correct proof of the

Chain rule. g differentiable at x , f differentiable at $y := g(x)$
 \Rightarrow the composition

$$(f \circ g)(x) := f(g(x))$$

is differentiable at x , with

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

□ By definition of the two derivatives,

$$\begin{aligned}
 g(x+h) &= g(x) + g'(x)h + \underbrace{s(h)}_0 h \\
 &\rightarrow 0 \text{ as } h \rightarrow 0
 \end{aligned}$$

$$\begin{aligned}
 f(y+k) &= f(y) + f'(y)k + \underbrace{r(k)}_0 k \\
 &\rightarrow 0 \text{ as } k \rightarrow 0.
 \end{aligned}$$

So,

$$\begin{aligned}
 f(g(x+h)) &= f\left(\underbrace{g(x)}_y + \underbrace{g'(x)h + s(h)h}_{=: k}\right) \\
 &= f(y) + f'(y)(g'(x)h + s(h)h) \\
 &\quad + r(k)k \\
 &= f(y) + f'(y)g'(x)h + t(h)h
 \end{aligned}$$

with

$$\begin{aligned}
 t(h) &:= f'(y)s(h) + r(k)(g'(x) + s(h)) \\
 &\rightarrow 0 \text{ as } h \rightarrow 0, \\
 \text{since } k &= (g'(x) + s(h))h \rightarrow 0 \text{ as } h \rightarrow 0. \blacksquare
 \end{aligned}$$

Example. $f(x) = \sqrt{x}$, $x \geq 0$,

$$g(x) = x^2, x \text{ real},$$

$$\begin{aligned}
 f(g(x)) &= \sqrt{x^2} = |x| = x, \quad x > 0, \\
 &= -x, \quad x < 0.
 \end{aligned}$$

continuous for x real and differentiable except at $x=0$.

Note f not differentiable at $x=0$.

Clearly,

$$\begin{aligned}
 \frac{d}{dx} f(g(x)) &= 1, \quad x > 0 \\
 &= -1, \quad x < 0.
 \end{aligned}$$

Checking the chain rule gives

$$f'(g(x))g'(x) = \frac{1}{2\sqrt{x^2}} \cdot 2x = \frac{x}{|x|}, \quad x \neq 0$$

$$= 1, \quad x > 0,$$

$$= -1, \quad x < 0,$$

as proved! (What about $g \circ f$?)

Roller's theorem. $-\infty < a < b < +\infty$,
 f continuous on $[a, b]$, and
differentiable on (a, b) , with
 $f(a) = f(b) \implies f'(c) = 0$ for at
least one c in (a, b) .

□ Consider cases.

① $f(x) = k = \text{constant}$ in (a, b) .

By continuity of f on $[a, b]$,

$f(x) = k$ on $[a, b]$, and $f'(x) = 0$ on
 (a, b) , so $f'(c) = 0$ for all c in
 (a, b) . (f has a right derivative
at a , and a left derivative at b ,
both = 0!)

② $f(x) > f(a)$ for some x in (a, b)

Since there's one x in (a, b)

with $f(x) > f(a) = f(b)$ we can,

by the continuity of f on $[a, b]$,
 assume that x is such that $f(x)$
 is the maximum of f over $[a, b]$.

The maximizing x is in (a, b)
 because there's one such x
 with $f(x) > f(a) = f(b)$. So $f'(x)$ exists.
 This means that

$$f(x+h) = f(x) + f'(x)h + \underbrace{r(h)}_{\rightarrow 0, h \rightarrow 0} h$$

for all sufficiently small
 $h \neq 0$. If we did not have
 $f'(x) = 0$ then there would be
 values of $x+h$, arbitrarily close
 to x , with $f(x+h) \neq f(x) + f'(x)h$,
larger than $f(x)$. If $f'(x) > 0$
 take $h > 0$ and tiny; if $f'(x) < 0$
 take $h < 0$ and tiny. It'd be
 better to write

$$f(x+h) = f(x) + f'(x) \left(1 + \underbrace{\frac{r(h)}{f'(x)}}_{\rightarrow 0, h \rightarrow 0} \right) h$$

to see this; here $f'(x) \neq 0$. ■

2. $f(x) < f_{\max}$ -- \Rightarrow f is simil.

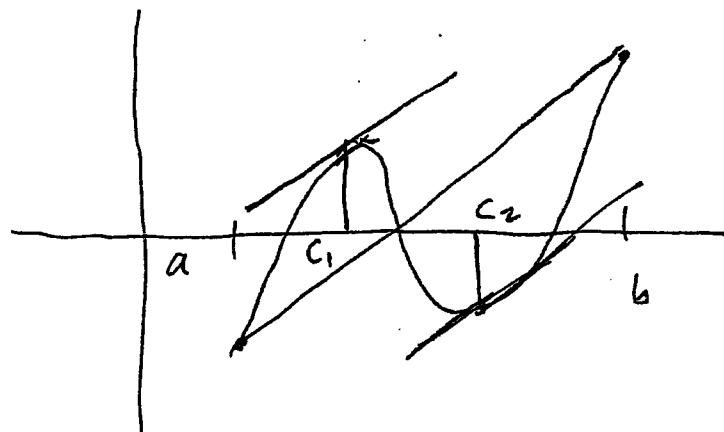
Mean Value Theorem for Derivatives

$-\infty < a < b < +\infty$, f continuous on $[a, b]$
differentiable on (a, b) \Rightarrow

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c in (a, b) .

This says that $f'(c)$ is the slope of the "secant line" connecting $(a, f(a))$ and $(b, f(b))$:



□. Apply Rolle's theorem to

$$F(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a} (x - a).$$

We have $F(a) = F(b) = 0$,

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} = 0$$

when $x = c$. ■

(I call this the "tilted Rolle theorem")

The result can be written

$$f(b) = f(a) + f'(c)(b-a)$$

or

$$f(x+h) = f(x) + f'(c)h,$$

with c between x and h. One nice way is to write

$$f(x+h) = f(x) + f'(x+\theta h)h,$$

$$0 < \theta < 1.$$

I call c "the elusive c ", since students are sometimes trying to find it. We don't find c !

The result simply allows us to bound the error in the approximation of $f(x+h)$ by $f(x)$ as

$$|f(x+h) - f(x)| = |f'(x+\theta h)h|$$

$$\leq \max_{0 \leq \theta \leq 1} |f'(x+\theta h)| |h|,$$

under the further assumption that $f'(x+\theta h)$ is continuous for $0 \leq \theta \leq 1$. Setting

$$M_1 := \max \{ |f'(x+\theta h)| : 0 \leq \theta \leq 1 \},$$

we have

$$|f(x+h) - f(x)| \leq M_1 |h| \quad (h \rightarrow 0)$$

This is normally written as

$$f(x+h) = f(x) + O(h), \quad h \rightarrow 0.$$

$M_1 = M_1(x, h)$ is a function of x and h , and of $f'(\cdot)$. We have

$$M_1 \rightarrow |f'(x)|, \quad h \rightarrow 0.$$

Cauchy Mean Value Theorem.

$-\infty < a < b < +\infty$, f and g continuous on $[a, b]$ and differentiable on (a, b) \Rightarrow

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

for some c with $a < c < b$.

□ Apply Rolle to

$$\begin{aligned} F(x) := & [f(b) - f(a)][g(x) - g(a)] - \\ & [g(b) - g(a)][f(x) - f(a)]. \end{aligned}$$

Corollary. If, in addition, $g'(x) \neq 0$ on (a, b) , then

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \quad a < c < b.$$

□ We must divide the result of the "CMVT" by $g'(c)[g(b)-g(a)]$. The factor $g'(c)$ is $\neq 0$ by explicit assumption. But also, by the "MVT", $g(b)-g(a) = g'(d)(b-a) \neq 0$, ($a < d < b$)! ■

Corollary (Johann Bernoulli's original theorem - not L'Hospital's!)

$-\infty < a < b < +\infty$, f and g differentiable on (a, b) , $g'(x) \neq 0$ on (a, b)

$$f(x) \rightarrow 0, g(x) \rightarrow 0, x \rightarrow a,$$

$$\frac{f'(x)}{g'(x)} \rightarrow l \text{ (real)}, x \rightarrow a$$

⇒

$$\frac{f(x)}{g(x)} \rightarrow l, x \rightarrow a.$$

□ x is in (a, b) if it's close to a .

Define $f(a) := g(a) := 0$; then f and g are continuous on $[a, x]$. The result follows from the "CMVT".

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}, a < c < x.$$

By assumption $\frac{f'(x)}{g'(x)} \rightarrow l$ as $x \rightarrow a$.

But $a < c < x \Rightarrow x \rightarrow a \Rightarrow c \rightarrow a \Rightarrow$

$$\frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)} \rightarrow l . \blacksquare$$

Examples. 1. $f(x) = \sin x \rightarrow 0, x \rightarrow 0$

$$g(x) = x \rightarrow 0, x \rightarrow 0$$

$$f'(x) = \cos x, g'(x) = 1 \neq 0,$$

$$\frac{f'(x)}{g'(x)} = \cos x \rightarrow 1, x \rightarrow 0 \Rightarrow$$

$$\frac{f(x)}{g(x)} = \frac{\sin x}{x} \rightarrow 1, x \rightarrow 0 .$$

Bernoulli's theorem was stated for right limits. Clearly it also works for left limits, and two sided limits. In this case $\frac{\sin x}{x}$ is even, that is it's a function of x^2 , so $x \rightarrow 0$ is a two-sided limit!

2. $f(x) = \ln(1+x), 1+x > 0$.

$$g(x) = x, f(x) \rightarrow 0, g(x) \rightarrow 0, x \rightarrow 0$$

$$f'(x) = \frac{1}{1+x}$$

$$g'(x) = 1 \neq 0$$

$$\frac{f'(x)}{g'(x)} \rightarrow 1, x \rightarrow 0 \Rightarrow$$

$$\frac{f(x)}{g(x)} = \frac{\ln(1+x)}{x} \rightarrow 1, x \rightarrow 0.$$

3.

$$\frac{f(x)}{g(x)} := \frac{e^x - 1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4}{x^5}$$

$$\sim \frac{0}{0}, x \rightarrow 0.$$

For short, $\frac{f(x)}{g(x)} \sim \frac{0}{0}, x \rightarrow a$ means
that $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ as $x \rightarrow a$.

By Cauchy MVT

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{f'(x_1)}{g'(x_1)}, \quad x_1 \text{ between } 0 \text{ and } x_0 := x \\ &= \frac{e^{x_1} - 1 - x_1 - \frac{1}{2}x_1^2 - \frac{1}{6}x_1^3}{5x_1^4} \sim \frac{0}{0}, x \rightarrow 0 \end{aligned}$$

By CMVT again

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{f'(x_1)}{g'(x_1)} = \frac{f''(x_2)}{g''(x_2)}, \quad x_2 \text{ between } 0 \text{ and } x_1 \\ &= \frac{e^{x_2} - 1 - x_2 - \frac{1}{2}x_2^2}{20x_2^3} \sim \frac{0}{0}, x \rightarrow 0 \end{aligned}$$

Holy Mackerel, Andy! Again,

$$\frac{f(x)}{g(x)} = \frac{f'(x_1)}{g'(x_1)} = \frac{f''(x_2)}{g''(x_2)} = \frac{f'''(x_3)}{g'''(x_3)} \quad x_3 \text{ between } 0 + x_2$$

$$= \frac{e^{x_3} - 1 - x_3}{60x_3^2} \sim \frac{0}{0}, x \rightarrow 0.$$

Holy Moley, Shazam, this is a job for Superman! Again,

$$\begin{aligned}\frac{f(x)}{g(x)} &= \frac{f'(x_1)}{g'(x_1)} = \dots = \frac{f^{(4)}(x_4)}{g^{(4)}(x_4)} \quad x_4 \text{ between } \\ &\qquad\qquad\qquad 0 \text{ and } x_3 \\ &= \frac{e^{x_4} - 1}{120x_4} \sim \frac{0}{0}, x \rightarrow 0\end{aligned}$$

Geeze, this is ridiculous! Or is it?

$$\begin{aligned}\frac{f(x)}{g(x)} &= \frac{f'(x_1)}{g'(x_1)} = \dots = \frac{f^{(5)}(x_5)}{g^{(5)}(x_5)} \quad x_5 \text{ between } \\ &\qquad\qquad\qquad 0 \text{ and } x_4 \\ &= \frac{e^{x_5}}{120} \rightarrow \frac{e^0}{120} = \frac{1}{6!}, x \rightarrow 0.\end{aligned}$$

When $x > 0$ we have

$$x = x_0 > x_1 > x_2 > x_3 > x_4 > x_5 > 0$$

so $x \rightarrow 0 \Rightarrow$ all $x_k \rightarrow 0$. Similarly if $x < 0$. Now set $c := x_5$ and rewrite the equation $f(x)/g(x) = e^c/120$:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{e^c x^{n+1}}{(n+1)!},$$

c between 0 and x , and strictly between if $x \neq 0$

We've generalized from $n=4$ to general n . Our next theorem will give an easy proof of a much more general result. For now let's play with this one. We show that

4. e is irrational.

□ With $x=1$ we have

$$e - \left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}\right) = \frac{e^n}{(n+1)!},$$

and $0 < e^n < 1$. Since $1 = e^0 < e^n < e^1 = e$ then

$$0 < \frac{1}{(n+1)!} < e - \left(1 + 1 + \dots + \frac{1}{n!}\right) < \frac{e}{(n+1)!}.$$

Suppose $e = \frac{p}{q}$ is rational:

p, q positive integers, $q \geq 1$.

For n large enough, $n!e$ is an integer, with

$$0 < \frac{1}{n+1} < n! \underbrace{\left[e - \left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}\right) \right]}_{\text{positive integer}} < \underbrace{\frac{e}{n+1}}_{n \rightarrow +\infty}$$

$\rightarrow 0,$

Nonsense! (Contradiction). So, e is irrational. ■

Taylor's theorem with derivative
(Lagrange) form of the remainder.

For fixed a and $x > a$, assume that $f^{(n+1)}(+)$ exists on (a, x) and $f^{(n)}(+)$ is continuous on $[a, x]$. Then

$$F(t) := \sum_{k=0}^n \frac{f^{(k)}(t)}{k!} (x-t)^k$$

is continuous on $[a, x]$ and differentiable on (a, x) with

$$F'(t) = \frac{f^{(n+1)}(t)}{n!} (x-t)^n.$$

Check this! It is the essence of the proof! Moreover,

$$F(x) = f(x)$$

and

$$F(a) = \boxed{\sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k := T_n(x)},$$

the Taylor polynomial of degree $\leq n$ for f at a . So, the error in the approximation of $f(x)$ by $T_n(x)$ is

$$E_n(x) := f(x) - T_n(x) = F(x) - F(a).$$

By the Cauchy Mean Value Theorem,

$$F(x) - F(a) = \frac{F'(c)}{G'(c)} [G(x) - G(a)], \quad a < c < x,$$

for any function $G(+)$, continuous on $[a, x]$ and differentiable on (a, x) with $G'(+) \neq 0$ there. Take

$$G(+) = (x-+)^{n+1}$$

to get

$$\boxed{f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}, \quad a < c < x}.$$

Similarly for $x < a$, except that (x, a) replaces (a, x) . Note that, here, the derivatives at $x=a$ only need to be right derivatives.

Problems. Take $a=0$. Find $T_n(x)$ and $E_n(x)$ for

1. $f(x) = e^x$, x real,
2. $f(x) = 1/(1-x)$, $x < 1$,
3. $f(x) = \ln(1+x)$, $x > -1$,
4. $f(x) = (1+x)^a$, $x > -1$,
5. $f(x) = \cos x$, x real
6. $f(x) = \sin x$, x real

7. $f(x) = \cosh x, x \text{ real}.$
8. $f(x) = \sinh x, x \text{ real}.$
9. $f(x) = \operatorname{atan} x, x \text{ real}.$
10. $f(x) = \operatorname{atanh} x, x \text{ real}.$
11. Check "the essence" of the Taylor-Lagrange theorem.

Some solutions et. al.

- ⑪ Each term, except the first, of

$$F(t) = f(t) + \sum_{k=1}^n \frac{f^{(k)}(t)}{k!} (x-t)^k,$$

is a product. Now

$$\begin{aligned} F'(t) &= f'(t) + \sum_{k=1}^n \frac{f^{(k)}(t)}{k!} k(x-t)^{k-1} (-1) \\ &\quad + \sum_{k=1}^n \frac{f^{(k+1)}(t)}{k!} (x-t)^k \\ &= \sum_{k=0}^n \frac{f^{(k+1)}(t)}{k!} (x-t)^k \\ &\quad - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!} (x-t)^k \\ &= \frac{f^{(n+1)}(t)}{n!} (x-t)^n \end{aligned}$$

Some wise person simplified a constant. L. d. . . .

①

$$f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \Rightarrow$$

$$c_n := \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

We get the result given earlier.

②

$$f(x) := \frac{1}{1-x}, \quad x < 1$$

$$f'(x) = \frac{(1-x)^0 - 1(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{(1-x)^2 0 - 1 \cdot 2(1-x)(-1)}{(1-x)^4} = \frac{2!}{(1-x)^3}$$

By induction

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \quad (*)$$

If true, then

$$\begin{aligned} f^{(n+1)}(x) &= \frac{(1-x)^{n+1} 0 - n!(n+1)(1-x)^{n+1}(-1)}{(1-x)^{2n+2}} \\ &= \frac{(n+1)!}{(1-x)^{(n+1)+1}} \end{aligned}$$

so $(*)$ is true for $n = 0, 1, 2, \dots$

$$c_n = \frac{f^{(n)}(0)}{n!} = 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \frac{x^{n+1}}{(1-c)^{n+2}},$$

$x < 1$, c between 0 and x .

We also have the exact result:

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x},$$

so the error is

$$\begin{aligned} \frac{1}{1-x} - (1 + x + x^2 + \dots + x^n) &= \frac{x^{n+1}}{1-x} \\ &= \frac{x^{n+1}}{(1-c)^{n+2}} \end{aligned}$$

and we can actually compute the elusive $c = c(x, n)$:

$$(1-c)^{n+2} = 1-x,$$

$$1-c = (1-x)^{\frac{1}{n+2}},$$

Note that $1-x$ and $1-c$ are > 0 .

$$c = 1 - (1-x)^{\frac{1}{n+2}},$$

for what it's worth (not much!). In this case $c = c(x, n)$ is a (single-valued) function of x and n . This is also true for the c in problem 1: $f(x) = e^x$.

③ $f(x) := \ln(1+x), x > -1$

$$f'(x) = \frac{1}{1+x}, f''(x) = \frac{(1+x)^0 - 1 \cdot 1}{(1+x)^2} = \frac{-1}{(1+x)^2},$$

$$f'''(x) = \frac{0 - (-1)^2(1+x)}{(1+x)^4} = \frac{2}{(1+x)^3}$$

Induction hypothesis (guess):

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \quad n=1,2,3,\dots$$

Proof. If so, then

$$\begin{aligned} f^{(n+1)}(x) &= \frac{0 - (-1)^{n-1}(n-1)! \cdot n (1+x)^{n-1}}{(1+x)^{2n}} \\ &= \frac{(-1)^n n!}{(1+x)^{n+1}}. \end{aligned}$$

Thus true for the stated n .

Taylor (Maclaurin since $a=0$) coefficients:

$$c_0 = \frac{f(0)}{0!} = 0$$

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{n}, \quad n=1,2,\dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n-1} x^n}{n}$$

$$+ \frac{(-1)^n x^{n+1}}{(n+1)(1+C)^{n+1}},$$

$x > -1$, C between 0 and x

For $x = 1$,

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n-1}}{n} + \\ + \frac{(-1)^n}{(n+1)(1+c)^n}, \quad 0 < c < 1.$$

So, since $1+c > 1$,

$$\left| \ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n-1}}{n} \right) \right| < \frac{1}{n+1} \\ \rightarrow 0, n \rightarrow +\infty$$

This is usually written as

$$\boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \ln 2}. \quad (*)$$

More about infinite series later.
We have

$\ln 2 \approx 0.6931471805599453$,
but this was not computed by
the incredibly slow technique
of (*)!

The arctangent is interesting-

⑨ $f(x) = \arctan x, x \text{ real.}$

We found that

$$f'(x) = \frac{1}{1+x^2},$$

but my attempt at finding

$f^{(n)}(x)$ by induction was "tedious".
 So I tried to "think" a little. Let
 $\xi = x^2$. We already know about
 $\frac{1}{1+\xi}$, when $\xi = -x$, and we have an
exact remainder for it:

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \frac{x^{n+1}}{1-x} .$$

Set $x = -\xi$:

$$\frac{1}{1+\xi} = 1 - \xi + \xi^2 - \dots + (-1)^n \xi^n + \frac{(-1)^{n+1} \xi^{n+1}}{1+\xi}$$

Set $\xi = t^2$:

$$f'(t) = \frac{1}{1+t^2} = 1 - t^2 + t^4 + \dots + (-1)^n t^{2n} + (-1)^{n+1} \frac{t^{2n+2}}{1+t^2} .$$

Now integrate between $t=0$ and $t=x$ to get

$$\begin{aligned} \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} \\ &\quad + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2} dt . \end{aligned}$$

Hence, if $x \neq 0$,

$$\begin{aligned} \left| \arctan x - \left(x - \frac{x^3}{3} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} \right) \right| &< \\ &< \int_0^x \frac{t^{2n+2}}{1+t^2} dt < \int_0^x t^{2n+2} dt = \frac{x^{2n+3}}{2n+3} . \end{aligned}$$

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Note that the functions here are all odd, so we would only need to compute them for $x > 0$. In any case

$$\left| \operatorname{atan} x - \left(x - \frac{x^3}{3} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} \right) \right| \leq$$

$$\leq \frac{|x|^{2n+3}}{2n+3} \leq \frac{1}{2n+3} \text{ for } |x| \leq 1$$

$$\rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Hence,

$$\boxed{\operatorname{atan} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}, \quad (*)$$

$|x| \leq 1.$

In fact we have

$$\boxed{\frac{\pi}{4} = \operatorname{atan} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots},$$

with very slow convergence!

However, "in principle" we could use $(*)$ to compute $\operatorname{atan} x$ for $|x| \leq 1$. What about $|x| > 1$?

Using basic integration techniques we have

$$\begin{aligned}
 \operatorname{atan}\left(\frac{1}{x}\right) &= \int_0^x \frac{dt}{1+t^2} & t &= \frac{1}{s} \\
 &= \int_{\infty}^x \frac{-\frac{1}{s^2} ds}{1+s^2} & t = 0 \Leftrightarrow s = +\infty \\
 &= - \int_{\infty}^x \frac{ds}{1+s^2} & t = \frac{1}{x} \Leftrightarrow s = x \\
 &= \int_x^{\infty} \frac{dt}{1+t^2} \\
 &= \int_0^{\infty} \frac{dt}{1+t^2} - \int_0^x \frac{dt}{1+t^2} \\
 &= \underbrace{\operatorname{atan}(+\infty) - \operatorname{atan} x}_{\frac{\pi}{2}}
 \end{aligned}$$

Thus we have the identity

$$\boxed{\operatorname{atan} x + \operatorname{atan}\left(\frac{1}{x}\right) = \frac{\pi}{2}, \quad \text{all real } x}.$$

So, being able to compute $\operatorname{atan} x$ for $|x| \leq 1$ suffices (in some sense).

$$(4) \quad f(x) = (1+x)^a = e^{a \ln(1+x)}, \quad x > -1.$$

$$f'(x) = a(1+x)^{a-1},$$

$$f''(x) = a(a-1)(1+x)^{a-2},$$

By induction,

$$f^{(n)}(x) = a(a-1)\dots(a-n+1)(1+x)^{a-n}, \\ n=0,1,2,\dots$$

The MacLaurin coefficients are
binomial coefficients:

$$c_n = \frac{f^{(n)}(0)}{n!} = \binom{a}{n} := \frac{a(a-1)\dots(a-n+1)}{n!}.$$

We can also use Pochhammer notation to write them as

$$c_n = (-1)^n \frac{(-a)(-a+1)\dots(-a+n-1)}{n!} \\ = : (-1)^n \frac{(-a)_n}{n!}.$$

In general, the Pochhammer symbol, also called the ascending factorial, is

$$(a)_n := 1, n=0$$

$$:= a(a+1)\dots(a+n-1), n=1,2,3,\dots$$

We have $n! = (1)_n$. Important special cases of $(1+x)^a$ are $a=1$, $a=\frac{1}{2}$ and $a=-\frac{1}{2}$. When $a=n$ is an integer ≥ 0 then $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$ is (rather famous) polynomial of degree n .

Euler's constant (introduction to alternating series with terms decreasing to zero).

$$a_1 := 1, \quad a_2 := \int_1^2 \frac{dt}{t} = \ln 2 - \ln 1$$

$$a_3 := \frac{1}{2}, \quad a_4 := \int_2^3 \frac{dt}{t} = \ln 3 - \ln 2$$

:

$$a_{2n-1} = \frac{1}{n}, \quad a_{2n} = \int_n^{n+1} \frac{dt}{t} = \ln(n+1) - \ln n$$

$$\rightarrow 0, \quad = \ln n + \ln\left(1 + \frac{1}{n}\right) - \ln n$$

 $n \rightarrow +\infty.$

$$= \ln\left(1 + \frac{1}{n}\right) \rightarrow 0, \quad n \rightarrow +\infty.$$

Claim: $1 = a_1 > a_2 > a_3 > \dots > a_n > 0$.

□ By MacLaurin-Lagrange,

$$\ln(1+x) = x - \frac{x^2}{2(1+c)^2}, \quad x > -1,$$

c between 0 and x .

In particular,

$$\ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2(1+c_n)^2},$$

$$0 < c_n < 1.$$

Hence

$$a_{2n-1} - a_{2n} = \frac{1}{n} - \ln\left(1 + \frac{1}{n}\right)$$

$$= \frac{1}{2n^2(1+c_n)^2} > 0$$

and

$$\begin{aligned} a_{2n} - a_{2n+1} &= \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} = \dots \\ &= \frac{\left(1 - \frac{1}{n}\right) + 2c_n(2+c_n)}{2n(n+1)(1+c_n)} \\ &> 0 \text{ since } c_n > 0. \end{aligned}$$

Partial sums:

$$S_n := a_1 - a_2 + a_3 - \dots + (-1)^{n-1} a_n$$

Since $a_n \downarrow 0$, clearly,

$$0 = S_0 < S_2 < S_4 < \dots < S_5 < S_3 < S_1,$$

$$S_{2n} \nearrow \gamma, S_{2n+1} \searrow \gamma \text{ as } n \rightarrow +\infty,$$

with

$$0 < \gamma - S_{2n} < S_{2n+1} - S_{2n} = \frac{1}{n+1} \rightarrow 0.$$

So,

$$S_{2n} = a_1 - a_2 + \dots + (-1)^{2n-1} a_{2n}$$

$$= (a_1 + a_3 + \dots + a_{2n-1}) -$$

$$-(a_2 + a_4 + \dots + a_{2n})$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - 1_{nn}$$

$$\rightarrow \gamma \doteq 0.577215669,$$

which is Euler's constant. Thus $\pi > e > \ln 2 > \gamma$ (at least empirically!).